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Unsteady Incompressible Inviscid Aerodynamics of a Thin Airfoil with Narrow Gaps

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Nonsteady load distributions are investigated on a two-dimensional thin airfoil having narrow gaps in an inviscid incompressible flow. First, the problem is solved numerically through the Possio's integral equation to yield a basis of comparison with the analytical method. The logarithmic singularity in the kernel function requires a special treatment for highly accurate results when the reduced frequency is large (say, more than 2). Second, under the restriction of very narrow gaps the problem is attacked analytically through the method of matched asymptotic expansion (MAE), which results in simple closed-form expressions. Comparisons of some results obtained from these two methods show excellent agreement with each other.

Nomenclature

$C(k)$	$=H_1^{(2)}(k)/[H_1^{(2)}(k)+iH_0^{(2)}(k)]$ Theodorsen's circulation function
$Ci(s)$	$=-\int_s^\infty t^{-1}\cos t\mathrm{d}t$
E_T	= relative error index
F	$=[\partial P_R/\partial\xi]_{\xi G}$
K	= matrix obtained from the kernel function through discretization
k	= reduced frequency ($=\omega/U$) (note that the half-chord is assumed to be unity)
$\hat{\ell}$	$=(L-L_R)/\epsilon L_R$
L	= lift force
$\hat{m}$	$=(\hat{M}_{1/2}-\hat{M}_R)/\epsilon\hat{M}_R$
$M_{1/2}$	= nose-down pitching moment about midchord
P	$=p_u-p_l$ loading distribution, downward positive
$\boldsymbol{P}$	$=-p/\rho U$
Q	= parameter concerning flow through the gap
$Si(s)$	$=\int_0^s t^{-1}\sin t\mathrm{d}t$
U	= freestream velocity
u,w	$=\xi$ and ζ direction components of perturbation velocity, respectively
γ	= Euler's constant
ξ,ζ	= Cartesian coordinates normalized by the half-chord length (Fig. 1)
$\bar{\xi},\bar{\zeta}$	= inner coordinates
ξ_G	$=\xi$ coordinate of the gap center
δ	= Dirac's delta function
ϵ	= half-width of gap, normalized by the half-chord length
ω	= angular frequency
ϕ	= perturbation velocity potential
ρ	= air mass density
$(\ )_a$	= behind the gap
$(\ )_f$	= in front of the gap
$(\ )^c$	= composite solution
$(\ )_G$	= quantities including gap effect
$(\ )^i,(\ )^{io}$	= inner solution and its outer limit
$(\ )^o,(\ )^{oi}$	= outer solution and its inner limit
$(\ )_R$	= quantities when there is no gap
$(\ )_{u,l}$	= upper and lower surfaces of airfoil
$\Delta(\ )$	= change due to gap effect
$(\hat{\ })$	= complex amplitude

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Introduction

THERE are many published works on the steady aerodynamics of airfoils having gaps, but rather few on unsteady cases. Of special interest among the many approaches for steady cases is one presented by White and Landahl,[1] since it yields simple explicit expressions having useful physical meanings.

The present work was undertaken in order to extend the idea of Ref. 1 to unsteady cases. A method similar to that of Ref. 1 is followed, except that pressure matching is used rather than the streamwise velocity matching. Also, it is necessary to overcome a difficulty stemming from a nonlinearity in the inner solution. The results obtained should be compared with other theories or experiments. For this purpose a numerical analysis by the so-called lattice method is also carried out; the Possio's integral equation is discretized and reduced to a set of simultaneous linear equations. For a large-frequency parameter (say, $k\geq 2$) satisfactory accuracy requires special treatment of the logarithmic singularity involved in the kernel function.[3] Each part of the chord in front of and behind the gap is divided into many small intervals by a semicircle. Both the analytical and numerical methods give results very close to each other, provided that the product $k\times\epsilon$ is small (ϵ is the ratio of gap to chord length).

In Appendix A, a simplified analytical method is presented. It imposes the Kutta condition at the front end of the gap where the inner solution is not required. Figure 1 shows the analytical model and symbols.

Numerical Analysis

Possio's Integral Equation for Incompressible Flow

For a thin airfoil executing simple harmonic motion in an incompressible inviscid flow of stream velocity U, the following well-known Possio's integral equation becomes useful

$$\hat{w}(x)=\frac{k}{2\pi}\int_{-1}^{1}\hat{P}(\xi)\left[\frac{1}{s}-ie^{-is}\int_{-\infty}^{s}\frac{e^{i\lambda}}{\lambda}\mathrm{d}\lambda\right]\mathrm{d}\xi \tag{1}$$

where $s=k(x-\xi)$. The semichord length is assumed to be unity. In many textbooks (for example, Ref. 2) the λ integral in Eq. (1) is written as

$$Ci(s)+i[Si(s)+\pi/2] \tag{2}$$

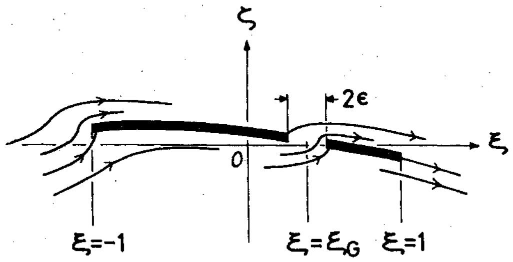

Fig. 1 Analytical model and symbols.

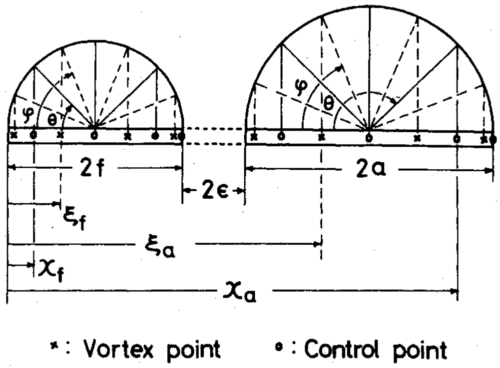

Fig. 2 Symbols for the numerical method (2-SCM).

But the more exact expression of Eq. (2) should be[3]

$$Ci(|s|)+i[Si(s)+\pi/2] \tag{3}$$

In addition $Ci(|s|)$ includes a logarithmic singularity

$$Ci(|s|)=\gamma+\log|s|+\int_0^{|s|} t^{-1}(\cos t-1)\,\mathrm{d}t \tag{4}$$

where γ is the Euler's constant. It has been found[3] that a careful treatment of the logarithmic singularity drastically improves the numerical accuracy for high frequency, say, $k\geq 2$. This improved treatment enforces the semicircle partitioning of the whole chord length.

Vortex Lattice Method in Presence of a Gap

Now we may consider a system of tandem airfoils. The clearance gap between them is assumed to be arbitrary, see Fig. 2. We expect this procedure will yield an "almost exact solution" with which the later analytical approximate solution can be compared. Each chord length is individually partitioned by a semicircle. The integral equation (1) may be discretized into the following form

$$\left\{\begin{array}{c}\hat{w}_f\\ \hline \hat{w}_a\end{array}\right\}=\left[\begin{array}{c|c}K_{ff} & K_{fa}\\ \hline K_{af} & K_{aa}\end{array}\right]\left\{\begin{array}{c}\hat{P}_f\\ \hline \hat{P}_a\end{array}\right\} \tag{5}$$

The matrices K_{ff} and K_{aa} contain the special countermeasure for the logarithmic singularity as contrived in a single airfoil,[3] while K_{af} and K_{fa} do not. This is because the logarithmic singularity becomes serious at or near diagonal elements of the K matrix. K_{af} and K_{fa} matrices have only a few elements near the diagonal and these are at the corners. This concept has been proved to be sound through many numerical examples as expected. The results show excellent agreement with those from the analytical method for very small ϵ.

Analytical Method: Matched Asymptotic Expansion (MAE)

Schwarz' Expression

For a thin airfoil having no gap, the loading is expressed by Schwarz[2] as follows

$$\hat{P}(\xi)=\frac{2}{\pi}[1-C(k)]\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\xi_1}{1-\xi_1}}\,\hat{w}(\xi_1)\mathrm{d}\xi_1 + \frac{2}{\pi}\int_{-1}^{1}\left[\sqrt{\frac{1-\xi}{1+\xi}}\sqrt{\frac{1+\xi_1}{1-\xi_1}}\frac{1}{\xi-\xi_1}-ik\Lambda_1(\xi,\xi_1)\right]\hat{w}(\xi_1)\mathrm{d}\xi_1 \tag{6}$$

where

$$\Lambda_1(\xi,\xi_1)\equiv \ell n\left|\frac{1-\xi\xi_1+\sqrt{1-\xi^2}\sqrt{1-\xi_1^2}}{\xi-\xi_1}\right| \tag{7}$$

If the thin airfoil has a very narrow gap, the first approximation of the loading may result from Eq. (6), using

$$\hat{w}(\xi_1)=\hat{w}_R(\xi_1)+\hat{Q}\epsilon\delta(\xi-\xi_G) \tag{8}$$

where $\hat{Q}$ is a constant to be determined. Here the effect of gap is represented by one due to a point source.

P Matching or *u* Matching?

For the steady case White and Landahl[1] used u matching, which brings about the relation

$$Q\sim u_R(\xi_G) \tag{9}$$

This is correct for a steady case; however, it is incorrect for an unsteady case. To justify this statement let us consider a fictitious case: replace the wake vortex sheet by an impermeable, fully flexible sheet. However, this may not alter the situation. This impermeable flexible sheet could not be distinguished from the original flexible airfoil which is also impermeable and may contain some zero-lift domain. Across this sheet there may be some jump of the u_R values for unsteady cases. If Eq. (9) were correct, some air would pass through an arbitrary "hole" made on the sheet (=wake)! This ridiculous conclusion is sufficient to negate the u matching, and suggests the correct formulation should be

$$\hat{Q}\sim\hat{P}_R(\xi_G) \tag{10}$$

which results from the P matching as shown later.

The inner solution must be valid at or near the gap where the disturbances could be large and the usual linearization could fail. Fortunately this will be seen to be an unnecessary anxiety later, at least for matching purposes.

Outer Solution

In constructing the outer solution, a small gap may be approximated by a point source for the upper surface flowfield. Therefore the normal wash should be expressed by Eq. (8), which is substituted into Eq. (6). Then we have the following two-term outer solution

$$\hat{P}^o(\xi)=\hat{P}_R(\xi)+\frac{2}{\pi}\hat{Q}\epsilon\left\{[1-C(k)]\sqrt{\frac{1-\xi}{1+\xi}}\sqrt{\frac{1+\xi_G}{1-\xi_G}} + \frac{1}{\xi-\xi_G}\sqrt{\frac{1-\xi}{1+\xi}}\sqrt{\frac{1+\xi_G}{1-\xi_G}}-ik\Lambda_1(\xi,\xi_G)\right\}+0(\epsilon^2,k\epsilon^2) \tag{11}$$

Substitute $\xi=\xi_G+\epsilon\bar{\xi}$ into Eq. (11) and then expand it for small ϵ to give the two-term inner limit

$$\hat{P}^{oi} = \hat{P}_R(\xi_G) + \epsilon\xi\hat{F}(\xi_G,k) + \frac{2}{\pi}\hat{Q}\left\{\frac{1}{\xi} + \epsilon\left[(1-C(k)) - \frac{1}{1-\xi_G^2} + ik\ell n\left|\frac{\epsilon\xi}{2(1-\xi_G^2)}\right|\right]\right\} + 0(\epsilon^2,k\epsilon^2) \quad (12)$$

where

$$\hat{F}(\xi_G,k) \equiv \frac{\partial}{\partial\xi}\hat{P}_R(\xi)\Big|_{\xi=\xi_G}$$

The Inner Solution

The inner solution desired should be exact in the neighborhood of the gap, so the linearized form might not be permitted. Thus we must use

$$p^i = -\rho/2[2U(u_u-u_l) + (u_u^2-u_l^2) + (w_u^2-w_l^2)] - \rho(\partial\phi_u/\partial t - \partial\phi_l/\partial t)] \quad (13)$$

However the matching procedure is carried out in the region of $|\xi|>1$ over the airfoil surface where $u_u = -u_l$, $w_u = +w_l$, and $\phi_u = -\phi_l$, since the airfoil considered has vanishing thickness. So Eq. (13) reduces to the same form as the linearized one

$$P^i = -2\rho(Uu^i + \partial\phi^i/\partial t)_u \quad (14)$$

or

$$\hat{P}^i = 2[ik\hat{\phi}^i + \epsilon^{-1}\partial\hat{\phi}^i/\partial\xi]_u \quad (15)$$

(Of course, this form is not valid for $|\xi| \leq 1$.) Let the disturbance velocity potential ϕ^i in the inner region be the following form

$$\phi^i(\xi,\zeta,t) = C(t) + B(t)\epsilon\xi + \epsilon\Phi_1^i(\xi,\zeta,t) \quad (16)$$

The first two terms in the right-hand side are suggested by considering the case where there is no gap, viz.,

$$u_R(t)\xi + f(t) \quad (17)$$

Taking account of a simple harmonic motion, and substituting Eq. (16) into Eq. (15), we have

$$\hat{P}^i = 2[\partial\hat{\Phi}_1^i/\partial\xi + \hat{B} + ik\hat{C} + ik\epsilon(\hat{B}\xi + \hat{\Phi}_1^i)]_u \quad (18)$$

Since the inner domain is quite local the inner flowfield may be essentially the same form as in the steady case, the time variable t being contained as a parameter. Hence following the similar line of Ref. 1, let us adopt

$$(\hat{\Phi}_1^i)_u = \hat{A}\ell n|\xi + \mathrm{sgn}(\xi)\sqrt{\xi^2-1}| + \hat{A}\,\mathrm{sgn}(\xi)\sqrt{\xi^2-1} + \hat{D} \quad (19)$$

$$(\partial\hat{\Phi}_1^i/\partial\xi)_u = \hat{A}\sqrt{|(\xi+1)/(\xi-1)|} \quad (20)$$

It should be noted that Eqs. (19) and (20) are real equations, and so the factor sgn(ξ) is necessary. For simplicity, however, sgn(ξ) is omitted hereafter. Use of Eqs. (19) and (20) in Eq. (18) yields

$$\hat{P}^i = 2\{\hat{A}\sqrt{|(\xi+1)/(\xi-1)|} + \hat{C}_1 + ik\epsilon[\hat{A}\ell n|\xi + \sqrt{\xi^2-1}| + \hat{A}\sqrt{\xi^2-1} + \hat{B}\xi + \hat{D}]\} + 0(\epsilon^2,k\epsilon^2) \quad (21)$$

where $\hat{C}_1 \equiv \hat{B} + ik\hat{C}$. This is a two-term inner solution. $\hat{A}$, $\hat{B}$, $\hat{C}_1$, and $\hat{D}$ remain to be determined. First, the Kutta condition requires $\hat{P}^i(-1-0) = 0$ ($|\xi| \to 1+0$), which gives

$$\hat{C}_1 + ik\epsilon(\hat{D} - \hat{B}) = 0 \quad (22)$$

Other conditions will be obtained through the matching procedure. The outer limit of Eq. (21) is written as

$$\hat{P}^{io} = 2\left\{\hat{A} + \hat{C}_1 + \hat{A}\frac{\epsilon}{\xi-\xi_G} + ik\epsilon\left[\hat{A}\ell n\left|\frac{2(\xi-\xi_G)}{\epsilon}\right| + \hat{D}\right] + ik(\hat{A}+\hat{B})(\xi-\xi_G)\right\} + 0(\epsilon^2,k\epsilon^2) \quad (23)$$

Matching

Comparing Eqs. (12) and (23) yields

$$2(\hat{A} + \hat{C}_1) = \hat{P}_R(\xi_G) \quad (24)$$

$$\hat{A} = \hat{Q}/\pi \quad (25)$$

$$ik(\hat{A}\ell n2 + \hat{D}) = \frac{\hat{Q}}{\pi}\left[1 - C(k) - \frac{1}{1-\xi_G^2} + ik\ell n\left|\frac{\epsilon}{2(1-\xi_G^2)}\right|\right] \quad (26)$$

$$2ik(\hat{A} + \hat{B}) = F(\xi_G,k) \quad (27)$$

Equations (22) and (24-27) determine $\hat{Q}$, $\hat{A}$, $\hat{B}$, $\hat{C}_1$, and $\hat{D}$. Let $\hat{Q}$ thus obtained be $\hat{Q}_1$ then

$$\hat{Q}_1 = \frac{\pi}{2}\,\frac{\hat{P}_R(\xi_G) - \epsilon F(\xi_G,k)}{1 - \epsilon\left[1 - C(k) - \dfrac{1}{1-\xi_G} + ik\left(\ell n\left|\dfrac{\epsilon}{4(1-\xi_G^2)}\right| + 1\right)\right]} \quad (28)$$

or retaining the zero-order term only,

$$\hat{Q}_2 \equiv \pi/2\hat{P}_R(\xi_G) + 0(\epsilon, k\epsilon\ell n\epsilon) \quad (29)$$

Equations (28) and (29) confirm Eq. (10) substantially.

Lift and Pitching Moment

Lift and pitching moment should be calculated using the composite solution $\hat{P}^c = \hat{P}^o + \hat{P}^i - \hat{P}^{io}$. However, elementary calculations show that the same results are obtainable using $\hat{P}^o$, if the desired accuracy is restricted to the order of $0(\epsilon)$; see Appendix B. Hence the lengthy expression of $\hat{P}^c$ is not shown here. The lift force is given by

$$\hat{L} \equiv \hat{L}_R + \Delta\hat{L} \equiv \hat{L}_R(1 + \epsilon\hat{l}) \quad (30)$$

where

$$\Delta\hat{L} = -2\hat{Q}_1\rho U\epsilon[C(k)\sqrt{(1+\xi_G)/(1-\xi_G)} + ik\sqrt{1-\xi_G^2}] \quad (31)$$

Similarly the pitching moment is

$$\hat{M}_{1/2} \equiv \hat{M}_R + \Delta\hat{M} \equiv \hat{M}_R(1 + \epsilon\hat{m}) \quad (32)$$

where

$$\Delta\hat{M} \equiv -\hat{Q}_1\rho U\epsilon\{[2\xi_G - 1 - C(k)]\sqrt{(1+\xi_G)/(1-\xi_G)} + ik\xi_G\sqrt{1-\xi_G^2}\} \quad (33)$$

Thus simple explicit formulas of a closed form are derived for an arbitrary mode of deformation and location of a gap, provided that the gap size is quite small. The exact applicable range depends on both k and ϵ (as shown later in Fig. 4). Moreover, it is noteworthy that when there is more than one gap simple superimposition will be applicable, provided every gap is quite small.

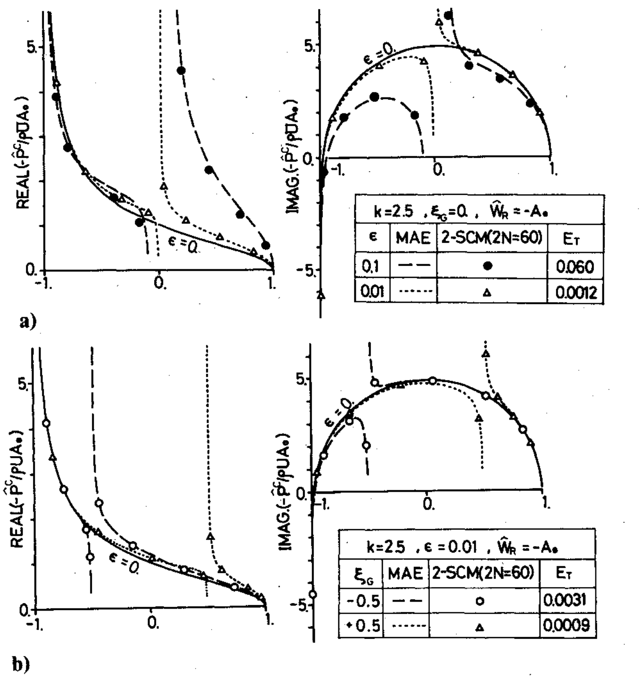

Fig. 3 Plunging motion, comparison of the analytical method (using $\hat{Q}_1$) with the numerical method: a) $k=2.5$, $\xi_G=0$, and $\epsilon=0.1$ and 0.01; b) $k=2.5$, $\epsilon=0.01$, and $\xi_G=\pm 0.5$.

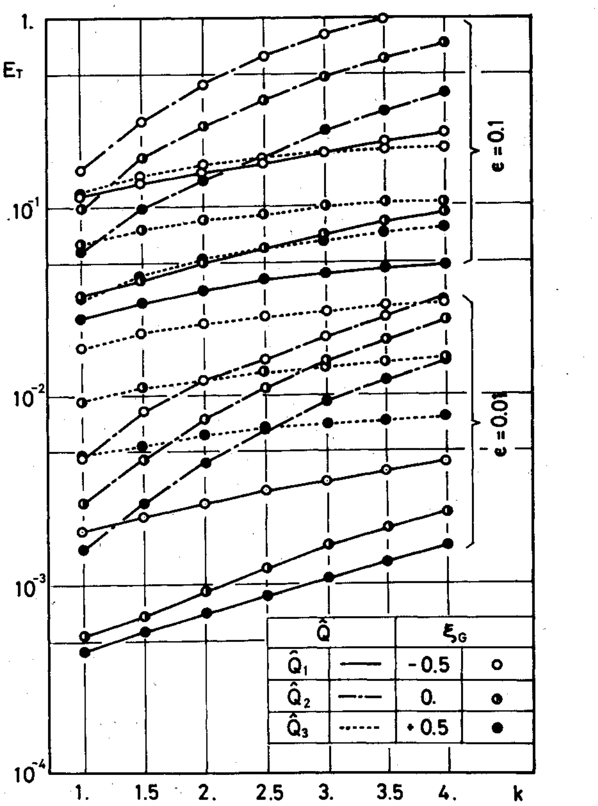

Fig. 4 Plunging motion error index between analytical and numerical results.

Result and Discussion

Although the present theory is applicable to general modes of deformation, the numerical results are given here only for plunging motion

$$\hat{w}_R(\xi) = -A_0 \tag{34}$$

Figure 3 shows loading distributions, which compare the numerical method (2-SCM) with the analytical method (MAE). In order to make the numerical results "almost

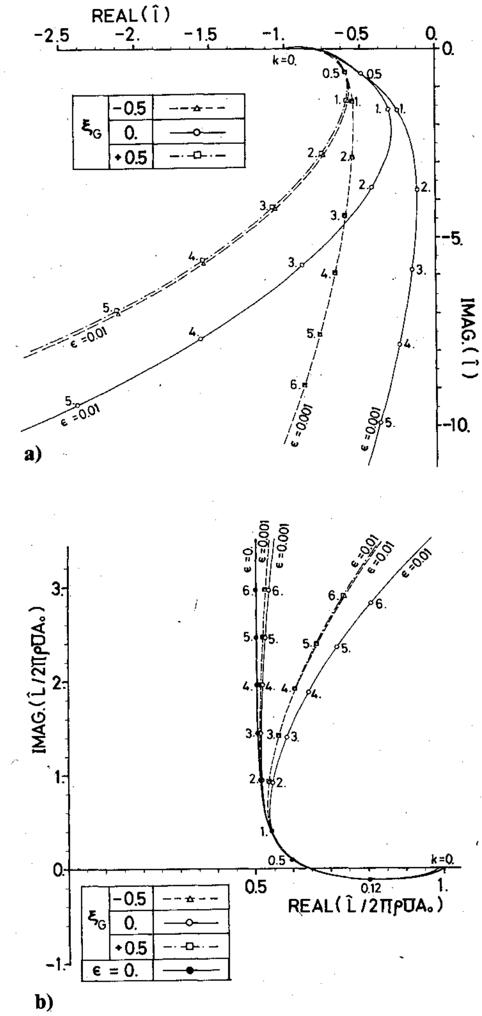

Fig. 5 Plunging motion, analytical method: a) function $\hat{l}$, using $\hat{Q}_1$; b) full lift force, using $\hat{Q}_1$.

exact," we select a rather large value of $2N=60$ as the total lattice number. In spite of a large value of $k=2.5$ excellent agreement is obtained, which confirms that both methods are quite sound, at least for $\epsilon \leq 0.05$. E_T has been introduced as a "relative error index" between both methods (see Appendix C). Figure 4 represents E_T vs k for $\epsilon=0.01$ and 0.1, $\xi_G=0$ and ± 0.5, using $\hat{Q}_1$, $\hat{Q}_2$, and $\hat{Q}_3$. It seems that E_T must be less than 0.05 to be of any useful purpose. As expected $\hat{Q}_1$ is best, while $\hat{Q}_2$ and $\hat{Q}_3$ are worse than $\hat{Q}_1$ by a factor of nearly 10. (For a definition of $\hat{Q}_3$, see Appendix A.) It is recognized that the much simpler expression of $\hat{Q}_2$ instead of $\hat{Q}_1$ may become applicable for, say, $\epsilon \leq 0.01$ and $k \leq 5$. Figure 5a shows the variation of the function $\hat{l}$ with k for $\xi_G=0$ and ± 0.5 and for $\epsilon=0.01$ and 0.001. $\hat{l}$ does not depend on ξ_G for $k=0$, the sign of ξ_G does somewhat affect l for all k. Figure 5b shows the variations of the full lift force due to k for $\epsilon=0$, 0.01, and 0.001 and for $\xi_G=0$ and ± 0.5. In addition to the discussions on $\hat{l}$, it is noteworthy that the absolute value of L becomes larger and the phase angle of L becomes smaller as ϵ increases from 0 to 0.01 for $k \geq 1.0$. But the physical reasoning of this is not yet well understood by us. Figure 6a shows variations of $\hat{m}$ due to k for $\epsilon=0.01$ and 0.001 and for $\xi_G=0$ and ± 0.5. Figure 6b shows variations of full pitching moment $M_{1/2}$. The gap positions ξ_G have little effect on $\hat{M}_{1/2}$ for $k \leq 1.0$. For larger k however the effect on both the absolute value and the phase angle of $\hat{M}_{1/2}$ becomes drastic. These somewhat peculiar results would be unimportant for the usual aeroelastic problems in aeronautics, because they appear only at much higher values of k. But it arouses our interest from the general scientific point of view.

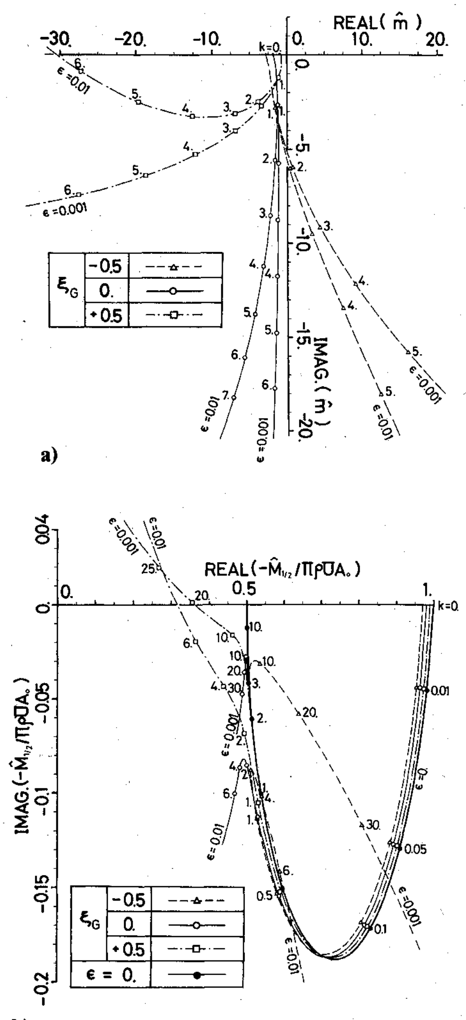

Fig. 6 Plunging motion, analytical method: a) function $\hat{m}$, using $\hat{Q}_1$; b) full moment $\hat{M}_{1/2}$, using $\hat{Q}_1$.

Conclusions

Theories are presented for a thin airfoil doing simple harmonic motions in an incompressible inviscid fluid. The analytical method utilizes the method of matched asymptotic expansion, applicable only when the product $k\epsilon$ (normalized gap size times reduced frequency) is small. The flow rate through the gap depends on the pressure difference at the gap station when the gap is blocked. The pressure distribution, lift, and pitching moment are expressed by simple explicit formulas. The numerical method, which is used as a basis for comparison with the analytical method, is based on the Possio's integral equation and is of course applicable to cases of arbitrary gap size. In order to obtain good accuracy for high frequency, the logarithmic singularity is appropriately treated using the two-semicircle method. Results from both methods agree quite well with each other for very small gap sizes.

Appendix A: Simplified Analytical Method

A simpe analytical method is also contrived which requires no inner solution. In order to determine $\hat{Q}$ in the outer solution [Eq. (11)], let us assume

$$\hat{P}^o(\xi_G - \epsilon) = 0 \tag{A1}$$

which may be considered as the Kutta condition applied at the front end of the gap. There is no logical basis for this idea, since the outer solution can give no details at or near the gap. Denoting $\hat{Q}$ obtained through Eq. (A1) as $\hat{Q}_3$, we have

$$\hat{Q}_3 = \frac{\pi}{2} \frac{\hat{P}_R(\xi_G) - \epsilon F(\xi_G, k)}{1 - \epsilon\left[1 - C(k) - \dfrac{1}{1-\xi_G^2} + ik\ell n\left|\dfrac{\epsilon}{2(1-\xi_G^2)}\right|\right]} \tag{A2}$$

which has a quite similar appearance to Eq. (28). Although the usefulness of $\hat{Q}_3$ is rather restricted, as seen from Fig. 4, it is interesting that $\hat{Q}_3$ is obtained easily without any inner solution.

Appendix B: Use of $\hat{P}^o$ Instead of $\hat{P}^c$

Strictly speaking, the lift force must be calculated through the composite solution $\hat{P}^c$ as follows

$$\hat{L}^c \equiv \left(\int_{-1}^{\xi_G - \epsilon} + \int_{\xi_G + \epsilon}^{1}\right)\hat{P}^c(\xi)\,d\xi \tag{B1}$$

Introduce another quantity $\hat{L}^o$ by

$$\hat{L}^o \equiv \int_{-1}^{1} \hat{P}^o(\xi)\,d\xi \tag{B2}$$

where the gap region is neglected. It is easily found through elementary but somewhat laborious integrations that

$$\hat{L}^c - \hat{L}^o = O(\epsilon^2) \tag{B3}$$

Hence we have used $\hat{L}^o$ as L.

Appendix C: Relative Error Index E_T

The relative error index E_T in this report is defined as follows

$$E_T \equiv \frac{\sum_{i=1}^{2N} \{\,|\mathrm{Re}(P_{Ai} - P_{Ni})| + |\mathrm{Im}(P_{Ai} - P_{Ni})|\,\}h_i}{\sum_{i=1}^{2N} \{\,|\mathrm{Re}P_{Ni}| + |\mathrm{Im}P_{Ni}|\,\}h_i} \tag{C1}$$

where

P_{Ai} = analytical airload at the ith station
P_{Ni} = numerical airload at the ith station
h_i = lattice length at the ith station
$2N$ = total lattice number of 2-SCM
Re,Im = real and imaginary parts, respectively

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